Turbulent free convection heat transfer to power-law fluids from arbitrary geometric configurations

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The problem of turbulent free convection heat transfer from curved surfaces to non-Newtonian power-law fluids has been investigated using the Nakayama–Koyama solution methodology. The scheme is designed to deal with bodies of arbitrary geometric configurations and hence can be viewed as a generalized version of the Shenoy–Mashelkar approach for turbulent free convection heat transfer from a flat vertical plate to a power-law fluid. The surface wall temperature is allowed to vary in the streamwise direction in an arbitrary fashion, and calculations are carried out for the turbulent free convection about the horizontal circular cylinder and sphere for illustrative purposes. Available theoretical and experimental data have been compared with the predictions of the present analysis and the comparison of results has been found to be reasonably good.

Keywords: turbulent convection; free convection; heat transfer; arbitrary shape; power-law fluids; non-Newtonian fluids

Introduction

For Newtonian fluids, turbulent free convection has been studied by a number of research workers, such as Colburn and Hougen,[1] Eckert and Jackson,[2] Bayley,[3] Fujii,[4] Kato et al.,[5] Cheesewright,[6] Kutateladze et al.,[7] Mason and Seban,[8] Papailiou and Lykoudis,[9] Cebeci and Kahttab,[10] Noto and Matsumoto,[11] Plumb and Kennedy,[12] Lin and Churchill,[13] George and Capp,[14] Thomas and Wood,[15] Ruckenstein and Felske,[16] Kawase and Ulbrecht,[17] and Nakayama and Koyama.[18] However, such extensive efforts do not exist for non-Newtonian fluids in turbulent free convection as can be seen from the comprehensive review articles by Shenoy[19,20] as well as Irvine and Karni.[21] The only existing analysis for power-law fluids are those of Shenoy and Mashelkar,[22] Ghosh et al.,[23] and Kawase.[24] All of them analyzed the turbulent natural convection heat transfer from a vertical plate under the assumption of high Prandtl numbers. Shenoy and Mashelkar[22] followed the integral approach of Eckert and Jackson,[2] while Ghosh et al.[23] used the eddy diffusivity expression from the model based on Levich's three zone concept. Kawase,[24] on the other hand, used the unified energy dissipation concept originally proposed by Calderbank and Moo-Young[25] for Newtonian fluids and extended the ideas for non-Newtonian flow.

The present paper extends the Shenoy–Mashelkar approach for the flat vertical plate to bodies of arbitrary geometric configurations using the solution method of Nakayama and Koyama,[18] who analyzed the turbulent free convection problem for Newtonian fluids.

Analysis

It is assumed that the geometric configuration has an arbitrary shape and the coordinate system is as shown in Figure 1. The body may be planar or axisymmetric, and its wall geometry is defined by the function $r(x)$. The wall surface is heated to $T_w(x)$ above the ambient temperature T_e, which is assumed to be constant (although the analysis can be easily extended to the variable T_e case as done by Nakayama et al.[26] for laminar free and forced convection). The flow is induced against the gravitational force g under the influence of the buoyancy force component parallel to the wall surface. The appearance of turbulence in the flow begins at the top of the surface and gradually extends to cover more and more of the surface as the Grashof number increases. Turbulence occurs when the surface in question is big or the temperature difference is large.

A usual control volume analysis within the boundary layer of thickness δ leads to the following integral forms of the momentum and energy equations under the Boussinesq's approximation on the buoyancy force:

$$\frac{d}{dx}\int_0^\delta r^*\rho u^2\,dy = r^*\rho\beta_0 g_x\int_0^\delta (T-T_e)\,dy - r^*\tau_w \qquad (1a)$$

$$\frac{d}{dx}\int_0^\delta r^*\rho u Cp(T-T_e)\,dy = r^* q_w \qquad (1b)$$

where

$$r^* = \begin{cases} 1: & \text{planar flow} \\ r(x): & \text{axisymmetric flow}\end{cases} \qquad (1c)$$

and

$$g_x = g\cos\phi = g\left\{1-\left(\frac{dr}{dx}\right)^2\right\}^{1/2} \qquad (1d)$$

In the above equations τ_w and q_w are the local wall shear and heat flux, while ρ, Cp, and β_0 are the density, specific heat, and thermal expansion coefficients, respectively. The tangential component of the acceleration due to gravity is indicated by g_x, which is related to the local surface orientation ϕ through Equation 1d. Moreover, the streamwise velocity and the local wall temperature are denoted by u and T with the subscripts e and w specifically pertaining to the boundary-layer edge and the wall surface, respectively.

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Theoretical analysis of free convection is normally more difficult than forced convection due to the coupling of the momentum and energy equation in the former case. When dealing with non-Newtonian fluids this task becomes even more formidable. Hence, certain simplifications are sought in order to facilitate a solution without sacrificing much accuracy.

The first step is to see how the equations would scale if the flow was purely forced convection. When dealing with turbulent non-Newtonian boundary-layer flow there is no denying that there exists a viscous sublayer that is very thin and close to the wall where the flow would be akin to a laminar forced convection flow. Also at the wall it is obvious that the local shear stress and the local heat flux assume their maximum value. Thus, an order of magnitude analysis of the kind used by Bejan[27] for Newtonian fluids can be used for the present case. This would follow the lines of Acrivos et al.[28] for laminar boundary-layer flows of non-Newtonian fluids past external surfaces. Thus,

$$\tau_w \sim \rho u_c^2 (\rho u_c l_c/\mu_{eff})^{-1/2} \tag{2a}$$

$$\frac{q_w}{(T_w - T_e)} \sim \frac{k}{l_c} \mathrm{Pr}_c^{1/3} (\rho u_c l_c/\mu_{eff})^{1/2} \qquad (\mathrm{Pr}_c > 1) \tag{2b}$$

where u_c is the characteristic velocity and Pr_c is the characteristic Prandtl number for non-Newtonian fluids defined on the lines of Acrivos et al.[28] and based on a characteristic length l_c as given below:

$$\mathrm{Pr}_c = Cp\mu_{eff}/k \tag{2c}$$

and μ_{eff} is defined as follows:

$$\mu_{eff} = \rho u_c l_c/(\rho u_c^{2-n} l_c^n/K)^{2/(n+1)} \tag{2d}$$

Combining Equations 2a and 2b gives the following:

$$\frac{q_w}{\rho Cp(T_w - T_e)u_c} \sim \frac{\tau_w}{\rho u_c^2} \mathrm{Pr}_c^{-2/3} \qquad (\mathrm{Pr}_c > 1) \tag{3}$$

It is now assumed that the above equation would hold even for the free convection flow if the characteristic velocity is related to the buoyancy rather than the free-stream velocity as in the forced convection case.

The dimensionless functions F and θ for the velocity and temperature profiles may be introduced as follows:

$$F(\eta) = u/u_c \quad \text{and} \quad \theta(\eta) = (T - T_e)/\Delta T \tag{4a,b}$$

Notation

a	Exponent of Grashof number in Equation 23
a'	Exponent of Grashof number in Equation 28a
b	Exponent of Prandtl number in Equation 23
b'	Exponent of Prandtl number in Equation 28a
B	Function of β, n as defined by Equation 13a
c'	Exponent of function in Equation 28a
C	Coefficient in Equation 23
Cp	Specific heat per unit mass
d	Diameter of cylinder or sphere
D	Function of β, n as defined by Equation 13b
f	Friction factor defined in Equation 6a
F	Function for velocity profile given by Equation 11a
g	Acceleration due to gravity
g_x	Tangential component of gravity defined by Equation 1d
Gr_x	Generalized local Grashof number for power-law fluids based on g_x and x and defined in Equation 17a
Gr_x^*	Generalized local Grashof number for power-law fluids based on g_x and x and defined in Equation 29a
Gr_d	Generalized Grashof number for power-law fluids based on g and d and defined in Equation 27a
h	Local heat transfer coefficient in Equation 18
h_{av}	Average heat transfer coefficient in Equation 26
i	Integer associated with the coordinate system in Equation 20a
I_t	Functions associated with deviation from unity and defined in Equations 17c and 21
j	Integer associated with the body shape given in Equation 20b
k	Thermal conductivity
K	Consistency index for a power-law fluid in Equation 7c
l_c	Characteristic length of scaling
L	Characteristic length of the geometric shape as used in Equation 24
m_t	Exponent associated with the wall temperature distribution as given by Equation 22
n	Pseudoplasticity index for a power-law fluid
Nu_x	Local Nusselt number defined by Equation 18
Nu_{dav}	Average Nusselt number based on diameter and defined by Equation 26
Nu_{Lav}	Average Nusselt number based on characteristic length and defined by Equation 24
Pr_x	Generalized local Prandtl number for power-law fluids based on g_x and x and defined in Equation 17b
Pr_x^*	Generalized local Prandtl number for power-law fluids based on g_x and x and defined in Equation 29b
Pr_d	Generalized Prandtl number for power-law fluids based on g and d and defined in Equation 27b
q	Function of β, n as defined by Equation 12
q_w	Heat flux at the wall
r^*	Function representing geometric configuration in Equation 1d
Ra_x	Rayleigh number defined as $\mathrm{Gr}_x \mathrm{Pr}_x$
Re_{gen}	Generalized Reynolds number in Equation 6a
T	Temperature
ΔT	Temperature difference defined by Equation 4c
T_e	Ambient temperature
T_w	Temperature of the body surface
u	Streamwise velocity component
u_c	Characteristic velocity in Equation 4a
x, y	Boundary-layer coordinates

Greek symbols

α, β	Dimensionless functions of n appearing in Equation 6a
β_0	Expansion coefficient of the fluid
γ_1	Coefficient defined by Equation 7c
δ	Boundary-layer thickness
η	Dimensionless variable defined by Equation 4d
θ	Dimensionless temperature profile defined by Equation 11b
μ	Viscosity of a Newtonian fluid
μ_{eff}	Effective viscosity defined in Equation 7a
ρ	Density of the fluid
τ_w	Local surface shear stress for power-law fluids defined in Equation 6b
ϕ	Local surface orientation in Equation 1d
Ω	Coefficient defined in Equation 7b

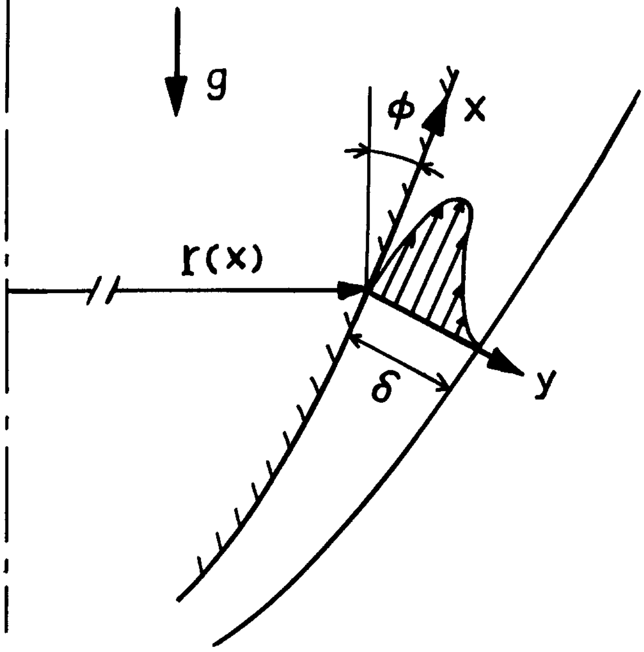

Figure 1 Physical model and its coordinate system

where

$$\Delta T = T_w - T_e \quad \text{and} \quad \eta = y/\delta \tag{4c,d}$$

The characteristic velocity u_c and the temperature difference ΔT are assumed to be functions of x.

Equations 1a and 1b can now be rewritten as follows:

$$A\frac{d}{dx}r^*\rho u_c^2\delta + r^*\tau_w = \mathrm{B}r^*\rho g_x\beta_0\,\Delta T\delta \tag{5a}$$

$$D\frac{d}{dx}r^*\rho u_c\,\Delta T\delta = r^*\tau_w\,\Delta T\mathrm{Pr}_c^{-2/3}/u_c \tag{5b}$$

where

$$A = \int_0^1 F^2 d\eta, \qquad B = \int_0^1 \theta\, d\eta, \qquad D = \int_0^1 F\theta\, d\eta \tag{5c,d,e}$$

Before solving the above equations, it is necessary to get expressions for τ_w and μ_{eff} for the free convection turbulent flow under consideration. For non-Newtonian power-law fluids, Dodge and Metzner[29] have provided a Blasius type of approximate equation for the friction factor in terms of the generalized Reynolds number as follows:

$$f = \alpha/\mathrm{Re}_{gen}^{\beta} \quad \text{for} \quad 5\times10^3 \leqq \mathrm{Re}_{gen} \leqq 10^5 \tag{6a}$$

where α and β are functions of n for the case of power-law fluids, and an explicit equation in f for the local surface shear stress can be obtained as follows:

$$\tau_w/\rho u_c^2 = 0.02332(\mu_{eff}/\rho u_c\delta)^{1/4} \tag{6b}$$

The effective viscosity, as introduced by Skelland,[30] can be written as

$$\mu_{eff} = (\Omega/0.02332)^4\gamma_1^{4\beta}\rho^{1-4\beta}\delta^{1-4\beta n}u_c^{1-4\beta(2-n)} \tag{7a}$$

where

$$\Omega = \alpha(0.817)^{2-\beta(2-n)}/2^{\beta n+1} \tag{7b}$$

and

$$\gamma_1 = 8^{n-1}K\{(3n+1)/4n\}^n \tag{7c}$$

Now an order of magnitude analysis of the Equations 5a and 5b can be carried out assuming x to be of the order of l_c.

Starting with the energy equation gives the following

$$\frac{\rho u_c\,\Delta T\delta}{l_c} \sim \rho u_c^2(\mu_{eff}/\rho u_c\delta)^{1/4}\,\Delta T\mathrm{Pr}_c^{-2/3}/u_c \tag{8a}$$

Convection Conduction

Solving the above yields

$$u_c \sim (l_c/\delta)^5(\mu_{eff}/\rho l_c)\mathrm{Pr}_c^{-8/3} \tag{8b}$$

Going to the momentum equation gives

$$\frac{\rho u_c^2\delta}{l_c} \quad \text{or} \quad \rho u_c^2(\mu_{eff}/\rho u_c\delta)^{1/4} \sim \rho g_x\beta_0\,\Delta T\delta \tag{8c}$$

Inertia Viscous Buoyancy

In the interplay of the above three forces, the buoyancy force is certainly the most important in the present circumstances because without it there would be no flow. However, it is worth establishing whether the boundary layer is governed by the inertia-buoyancy balance or the viscous-buoyancy balance. For this purpose, each of the expressions given in Equation 8c are divided by buoyancy scale $\rho g_x\beta_0\,\Delta T\delta$. Eliminating u_c using Equation 8b gives the following:

$$\left(\frac{l_c}{\delta}\right)^{10}\frac{1}{\mathrm{Gr}_c\mathrm{Pr}_c^{16/3}} \qquad \left(\frac{l_c}{\delta}\right)^{10}\frac{1}{\mathrm{Gr}_c\mathrm{Pr}_c^{14/3}} \qquad 1 \tag{8d}$$

Inertia Viscous Buoyancy

where

$$\mathrm{Gr}_c = \frac{\rho^2 l_c^3(g_x\beta_0\,\Delta T)}{\mu_{eff}^2} \tag{9a}$$

and

$$\mathrm{Pr}_c = Cp\mu_{eff}/k \tag{9b}$$

It is obvious from the above that the competition between inertia and viscous forces is essentially decided by the magnitude of the Prandtl number. From the definition of Pr_c, it is clear that for larger effective viscosities, as in the case of non-Newtonian fluids, the Prandtl number in turn would acquire values much greater than 1. Since the exponent of Prandtl number in the inertia scale is greater than that in the viscous scale, it would be quite proper to neglect the inertia term in comparison to the viscous term for the non-Newtonian fluids under consideration. This sort of scale holds for laminar as well as turbulent natural convection. Hence, it has been the commonly followed practice to neglect inertia terms in the study of natural convection as can be seen from the review articles of Shenoy.[19,20] Thus, under the high Prandtl number assumption, Equation 5a can be written as

$$\tau_w = B\rho g_x\beta_0\,\Delta T\delta \tag{10}$$

Equations 5b and 10 are the final simplified forms of the governing equations that are now to be solved. This requires expressions for the dimensionless velocity and temperature profiles. These are sought by following the arguments set forth by Eckert and Jackson.[2] They noted that in turbulent forced convection equations of the form $F(\eta) = \eta^{1/7}$ and $\theta(\eta) = 1 - \eta^{1/7}$ hold rather well. For turbulent free convection, they found that the experimental data could be fitted well with the same equation for temperature profile while the velocity profile needed to be modified to $F(\eta) = \eta^{1/7}(1-\eta)^4$. For power-law fluids, the velocity profile for turbulent forced convection flow can be taken as $F(\eta) = \eta^q$ where $q = \beta n/\{2 - \beta(2 - n)\}$ as given by Skelland.[30] In the present free convection case, the velocity and temperature profiles will be assumed by analogous arguments to those of Eckert and Jackson,[2] making use of the forced convection expression for power-law fluids as given by Skelland.[30] The dimensionless velocity and temperature profiles that are assumed to fit the turbulent free convection flow of power-law fluids are

$$F(\eta) = \eta^q(1 - \eta)^4 \tag{11a}$$

$$\theta(\eta) = 1 - \eta^q \tag{11b}$$

where

$$q = \beta n/\{2 - \beta(2 - n)\} \tag{12}$$

Using Equations 11a and b, the expressions for B and D as defined in Equations 5d and e can be easily obtained as

$$B = q/(q + 1) \tag{13a}$$

$$D = \frac{3}{q+1} - \frac{2}{q+2} + \frac{6}{q+3} - \frac{4}{q+4} + \frac{1}{q+5} - \frac{1}{2q+1} - \frac{6}{2q+3} - \frac{1}{2q+5} \tag{13b}$$

Note that for Newtonian fluids when q takes the value of 1/7, we have $B = 1/8$ and $D = 0.0366$, which are both identical to the values obtained by Nakayama and Koyama.[18] On combining Equations 6b and 10, the following expression for u_c is obtained:

$$u_c = (Bg_x\beta_0\,\Delta T/\Omega)^{\frac{1}{2-\beta(2-n)}}\delta^{\frac{1+\beta n}{2-\beta(2-n)}}(\rho/\gamma_1)^{\frac{\beta}{2-\beta(2-n)}} \tag{14}$$

This equation, along with Equation 7a, can now be substituted into Equation 5b to eliminate u_c and τ_w and results in the following equation after mathematical rearrangement of the terms.

$$\begin{aligned} &\frac{d}{dx}\delta^{\frac{2\{6-\beta(10-n)\}}{3\{2-\beta(2-n)\}}} + \delta^{\frac{2\{6-\beta(10-n)\}}{3\{2-\beta(2-n)\}}} \times \frac{d}{dx}\ln\left\{r^*\,\Delta T(g_x\,\Delta T)^{\frac{1}{2-\beta(2-n)}}\right\}^{\frac{2\{6-\beta(10-n)\}}{3\{3-2\beta(1-n)\}}} \\ &= \frac{2\{6-\beta(10-n)\}}{3\{3-2\beta(1-n)\}}\frac{\Omega}{D}\left(\frac{\Omega}{B}\right)^{\frac{2-5\beta(2-n)}{3\{2-\beta(2-n)\}}}\left(\frac{\Omega}{0.02332}\right)^{-8/3} \\ &\times\left(\frac{\rho Cp}{k}\right)^{-2/3}\left(\frac{\gamma_1}{\rho}\right)^{-\frac{8\beta}{3\{2-\beta(2-n)\}}}(g_x\beta_0\,\Delta T)^{-\frac{2-5\beta(2-n)}{3\{2-\beta(2-n)\}}} \end{aligned} \tag{15}$$

The solution to the above equation may readily be obtained as

$$\left(\frac{\delta}{x}\right)^{\frac{2\{6-\beta(10-n)\}}{3\{2-\beta(2-n)\}}} \mathrm{Gr}_x^{\frac{1-\beta(2-n)}{3\{2-\beta(2-n)\}}}\mathrm{Pr}_x^{2/3} = \frac{2\{6-\beta(10-n)\}}{3\{2-\beta(2-n)\}}\frac{\Omega}{D}\left(\frac{\Omega}{B}\right)^{\frac{2-5\beta(2-n)}{3\{2-\beta(2-n)\}}}\left(\frac{\Omega}{0.02332}\right)^{-8/3} I_t \tag{16}$$

where

$$\mathrm{Gr}_x = (\rho/\gamma_1)^{8\beta}x^{4\beta(2+n)}(g_x\beta_0\,\Delta T)^{4\beta(2-n)} \tag{17a}$$

$$\mathrm{Pr}_x = \frac{\rho Cp}{k}(\gamma_1/\rho)^{4\beta}x^{\{3-4\beta(2+n)\}/2}(g_x\beta_0\,\Delta T)^{\{1-4\beta(2-n)\}/2} \tag{17b}$$

and

$$I_t = \frac{\int_0^x\left\{r^{*2\{6-\beta(10-n)\}}g_x^{3+10\beta(1-n)}\,\Delta T^{15-2\beta(5+4n)}\right\}^{1/3\{3-2\beta(1-n)\}}dx}{x\left\{r^{*2\{6-\beta(10-n)\}}g_x^{3+10\beta(1-n)}\,\Delta T^{15-2\beta(5+4n)}\right\}^{1/3\{3-2\beta(1-n)\}}} \tag{17c}$$

The foregoing function I_t accounts for the total combined effects of arbitrary geometries and wall temperature distributions, while Equations 17a and 17b define the local Grashof and Prandtl numbers, respectively.

The local Nusselt number Nu_x is related to u_c and δ as

$$\begin{aligned}\mathrm{Nu}_x &= \frac{hx}{k} = \left(\frac{\rho Cpu_cx}{k}\right)\left(\frac{\tau_w}{\rho u_c^2}\right)\left(\frac{Cp\mu_{eff}}{k}\right)^{-2/3} \\ &= \left(\frac{\rho Cpu_cx}{k}\right)\Omega\left(\frac{\gamma_1}{\rho u_c^{2-n}\delta^n}\right)^{\beta} \\ &\cdot\left\{\frac{\rho Cp}{k}\left(\frac{\Omega}{0.02332}\right)^4\left(\frac{\gamma_1}{\rho}\right)^{4\beta}\delta^{1-4\beta n}u_c^{1-4\beta(2-n)}\right\}^{-2/3}\end{aligned} \tag{18}$$

u_c in the above equation may be eliminated in favor of δ, using Equation 14. Then, Equation 16 may be substituted into the equation. After considerable manipulations, one obtains the final Nu_x expression as follows:

$$\begin{aligned}\mathrm{Nu}_x &= (0.02332)^{\frac{4\{3-2\beta(1-n)\}}{6-\beta(10-n)}}\Omega^{\frac{-9}{6-\beta(10-n)}}B^{\frac{3+10\beta(1-n)}{2\{6-\beta(10-n)\}}} \\ &\cdot\left[\frac{2\{6-\beta(10-n)\}}{3\{3-2\beta(1-n)\}}\frac{I_t}{D}\right]^{\frac{-3+2\beta(7+2n)}{2\{6-\beta(10-n)\}}}\mathrm{Gr}_x^{\frac{21-4\beta(8+n)}{8\{6-\beta(10-n)\}}} \\ &\cdot\mathrm{Pr}_x^{\frac{3-\beta(8+n)}{6-\beta(10-n)}}\end{aligned} \tag{19}$$

Results and discussion

It is of great interest to investigate certain cases for which the function I_t remains constant. Any geometry near the stagnation point may be characterized by the following proportional relationship:

$$r^* \propto x^i \quad \text{where} \quad i = \begin{cases}0: & \text{plane body}\\ 1: & \text{axisymmetric body}\end{cases} \tag{20a}$$

$$g_x \propto x^j \quad \text{where} \quad j = \begin{cases}0: & \text{pointed body}\\ 1: & \text{blunt body}\end{cases} \tag{20b}$$

For example, integers (i,j) should be set to $(0,0)$ for a flat plate, $(1,0)$ for a vertical cone pointing downward, $(0,1)$ for the stagnation region on a horizontal circular cylinder, and $(1,1)$ for the stagnation region of a sphere. Equation 17c under the condition described by the foregoing proportional relationship yields

$$I_t = \left[1 + \frac{2\{6-\beta(10-n)\}i + \{3+10\beta(1-n)\}j + \{15-2\beta(5+4n)\}m_t}{3\{3-2\beta(1-n)\}}\right]^{-1} \tag{21}$$

where m_t is associated with the wall temperature distribution around the stagnation point, which is assumed to follow

$$\Delta T \propto x^{m_t} \tag{22}$$

The wall temperature distribution reflects on the function I_t in such a manner that I_t diminishes as T_w increases downstream (i.e., $m_t > 0$). Equation 21 for $n = 1$ and $\beta = 1/4$ yields Nakayama and Koyama's expression for Newtonian fluids.[18] Moreover, $I_t = 1$ for the case of an isothermal flat plate $(i,j,m_t) = (0,0,0)$. Thus, Equation 19 for Nu_x naturally reduces to the one derived by Shenoy and Mashelkar[22] who tabulated

Table 1 Exponents a and b

n	α	β	a	b
1.0	0.0790	0.250	0.400	0.200
0.9	0.0770	0.255	0.405	0.199
0.8	0.0760	0.263	0.410	0.192
0.7	0.0752	0.270	0.416	0.187
0.6	0.0740	0.281	0.422	0.174
0.5	0.0723	0.290	0.429	0.165
0.4	0.0710	0.307	0.438	0.138
0.3	0.0683	0.325	0.448	0.106
0.2	0.0646	0.349	0.463	0.054

Table 2 Coefficient C for isothermal bodies

n	Flat plate (0, 0)	Cone (1, 0)	Cylinder (0, 1)	Sphere (1, 1)
1.0	0.0402	0.0356	0.0380	0.0345
0.9	0.0428	0.0379	0.0402	0.0365
0.8	0.0443	0.0389	0.0412	0.0373
0.7	0.0450	0.0394	0.0415	0.0376
0.6	0.0464	0.0403	0.0422	0.0380
0.5	0.0477	0.0411	0.0427	0.0384
0.4	0.0483	0.0408	0.0420	0.0374
0.3	0.0497	0.0411	0.0416	0.0367
0.2	0.0501	0.0401	0.0392	0.0341

the values a, b, and C such that Nu_x can readily be evaluated from

$$\mathrm{Nu}_x = C\,\mathrm{Gr}_x^a\,\mathrm{Pr}_x^b \tag{23}$$

The values a, b, and C are furnished in Tables 1 and 2 for the isothermal flat plate and cone as well as the stagnation regions of isothermal cylinder and sphere.

The averaged Nusselt number, Nu_L, which is often more convenient to use for heat transfer estimation, can easily be derived by taking an integrated average over a length L.

$$\mathrm{Nu}_{Lav} = \frac{h_{av}L}{k} = \frac{L\int_0^L (\mathrm{Nu}_x/x)r^*\,dx}{\int_0^L r^*\,dx} = \frac{1+i}{i + \dfrac{3\{3-2\beta(1-n)\} + \{3+10\beta(1-n)\}(j+m_t)}{2\{6-\beta(10-n)\}}}\,\mathrm{Nu}_x\big|_{x=L} \tag{24}$$

For illustrative purposes, numerical integrations have been carried out, using Equation 19 along with 17c to find the peripheral variation of local heat transfer coefficient h and its averaged value h_{av} over an isothermal horizontal circular cylinder and sphere. The singularity at $x = 0$ can be removed by evaluating I_t according to Equation 21. The local heat transfer results obtained with $n = 1$, 0.5, and 0.3 are presented in Figures 2a and b for a cylinder and a sphere, respectively. The local heat transfer coefficient increases downstream, attains its maximum value at the upper half of the cylinder, and then decreases toward the rear stagnation point. A very similar heat transfer rate distribution prevails over the surface of the sphere. The location where the maximum heat transfer rate takes place, however, moves somewhat downstream for the case of the sphere. The peripheral distribution of heat transfer coefficient is rather sensitive to the power-law index n. An increase in n tends to smooth out the variation in the heat transfer coefficient

Table 3 Coefficient for averaged Nusselt number $\mathrm{Nu}_{dav}/\mathrm{Gr}_d^a\,\mathrm{Pr}_d^b$

n	Isothermal cylinder	Isothermal sphere
1.0	0.0292	0.0316
0.9	0.0304	0.0332
0.8	0.0306	0.0337
0.7	0.0302	0.0336
0.6	0.0300	0.0337
0.5	0.0298	0.0337
0.4	0.0284	0.0325
0.3	0.0273	0.0316
0.2	0.0249	0.0292

around the periphery. The numerical integration results on a cylinder and a sphere are listed in Table 3 in terms of

$$\mathrm{Nu}_{dav}/\mathrm{Gr}_d^a\,\mathrm{Pr}_d^b \tag{25}$$

where

$$\mathrm{Nu}_{dav} = h_{av}d/k \tag{26}$$

is the averaged Nusselt number based on the diameter d and

$$\mathrm{Gr}_d = (\rho/\gamma_1)^{8\beta} d^{4\beta(2+n)} (g\beta_0\,\Delta T)^{4\beta(2-n)} \tag{27a}$$

$$\mathrm{Pr}_d = \frac{\rho C p}{k} (\gamma_1/\rho)^{4\beta} d^{\{3-4\beta(2+n)\}/2} (g\beta_0\,\Delta T)^{\{1-4\beta(2-n)\}/2} \tag{27b}$$

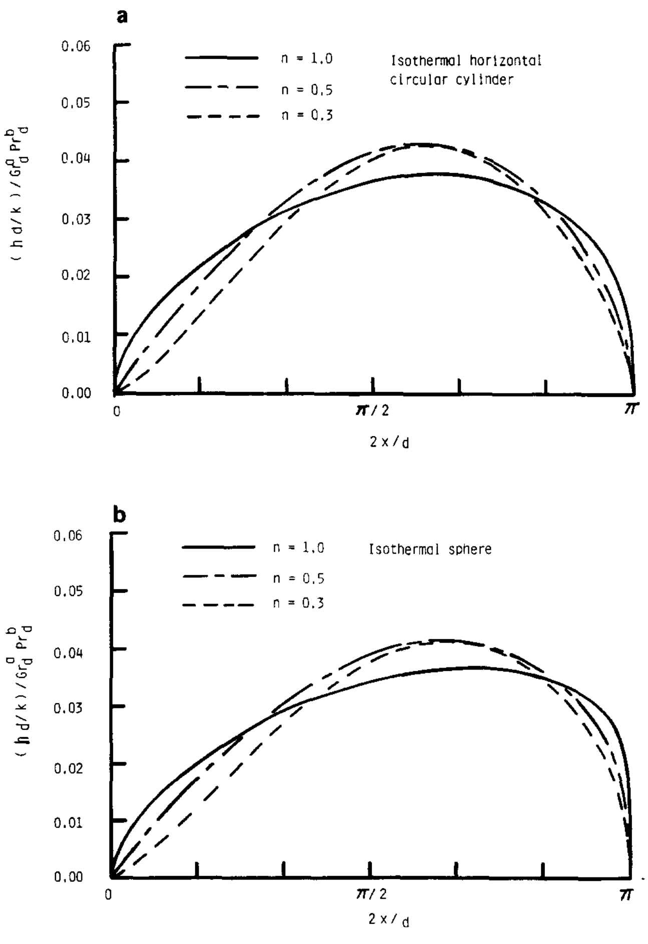

Figure 2 Local heat transfer results: (*a*) isothermal horizontal circular cylinder; (*b*) isothermal sphere

Comparison with existing theoretical analysis and experimental data

In the case of an analysis of the kind done previously, there is often skepticism about its reliability and utility because of the use of the integral method when modern computers are available and because of the assumptions made in order to facilitate a solution.

As regards the use of integral methods, there is often an impression that the method is insensitive to postulated temperature and velocity distributions and erroneous results are obtained. This impression is not truly correct. In fact, integral methods provide the easiest way of determining heat transfer coefficients with considerable accuracy as will be shown later when a comparison of results is made with the available numerical scheme. This is because the integral method, though rather insensitive to postulated temperature and velocity distributions, predicts the thermal boundary layer thickness and its dependence on the Grashof and Prandtl numbers very accurately and hence gives the right measures of the heat transfer coefficient. When dealing with non-Newtonian boundary-layer problems, in spite of the advent of powerful personal computers and efficient numerical algorithms, the integral method is still popular as has been shown by Nakayama[31] for the following reasons:

(1) integral formulation of governing equations is very simple and often analytical solutions become possible;
(2) integral solutions are successful in providing critical parameters that are normally essential in engineering applications;
(3) effects of various parameters such as differences in rheological behavior can be easily gotten from the integral solution for comparison without having to rerun the scheme as in numerical algorithms.

Further, from a practical viewpoint, sophistications in an analysis would be justified only when such sophistications lead to a substantial improvement in its accuracy or applicability to other physical situations. With all the available sophistication, no one has yet attempted to study the turbulent free convection problem from any arbitrary geometry other than the simplest vertical flat plate. Using the integral method, the present analysis could achieve a solution that, apart from being simple, is applicable to arbitrary shapes.

Now in order to determine whether the assumptions made during the problem analysis are correct, a comparison between the obtained solutions and existing theoretical solutions and experimental data is done.

Newtonian fluids

Experimental data and numerical simulation results are available based on a turbulence model for turbulent free convection from a vertical flat plate to Newtonian fluids of moderately high Prandtl number. Fujii et al.[32] carried out experiments using water and spindle oil, and their results for moderately high Prandtl numbers are in reasonable agreement with the present analysis derived under the high Prandtl number approximation as shown in Figures 3a and b. It is worth noting that the present solution predicts a slight decrease in the Nusselt number for increasing Prandtl number at fixed values of Rayleigh number $Ra_x = Gr_x Pr_x$. Though this point has not been highlighted by previous workers, it can be seen that experimental data of Fujii et al.,[32] in fact, show a possible decrease in the Nusselt number for increasing Prandtl number as can be seen by comparing the data shown in Figures 3a and b. The dependence of the Nusselt number on Prandtl number is not very strong, but it is certainly consistent with the expressions proposed in the present analysis.

In the same figures, the numerical simulation results obtained by Lin and Churchill[13] using the k-ε model of Jones and Launder[33] are plotted. Whereas the present analysis slightly overpredicts the experimental data, the results of Lin and Churchill[13] slightly underpredict it, especially, in Figure 3b, which is for a moderately high Prandtl number. Though the analysis of Lin and Churchill[13] is based on a sophisticated full numerical simulation, it is not devoid of approximations and assumptions. In fact, they found that even an elaborate two-equation turbulence model like the k-ε model was by no means readily applicable for the prediction of turbulent free convection. Since eddy diffusivities depend on both the momentum and energy equations, the existing models tuned for forced convection had to be modified to account for such dependency. Thus, Lin and Churchill[13] had to use empirical coefficients evaluated for forced convection and introduce an additional term for turbulent kinetic energy production due to buoyancy. This term was varied from 0 to 1 to match the experimental data in the range of $0.7 \leqq Pr \leqq 58$.

The discussion in the previous paragraph reemphasizes the utility of the simple integral method for solving problems like the one under consideration. Despite the simplicity of the integral method, it can be seen that the accuracy is not truly sacrificed.

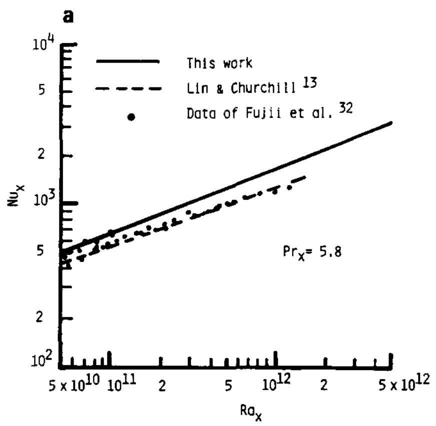

Figure 3(a) Comparison of the local Nusselt number from the present work for $Pr_x = 5.8$ with experimental data of Fujii et al.[32] for water and the theoretical predictions of Lin and Churchill[13] numerical solution for Newtonian fluids

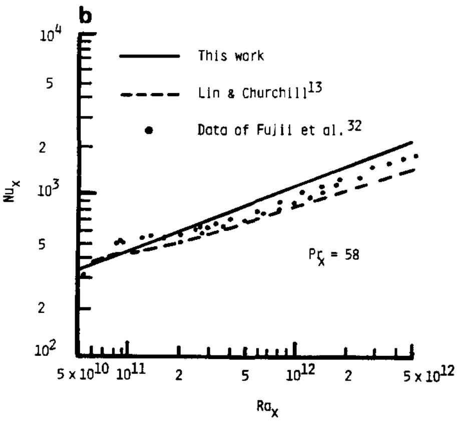

Figure 3(b) Comparison of the local Nusselt number from the present work for $Pr_x = 58$ with experimental data of Fujii et al.[32] for spindle oil and the theoretical predictions of Lin and Churchill[13] numerical solution for Newtonian fluids

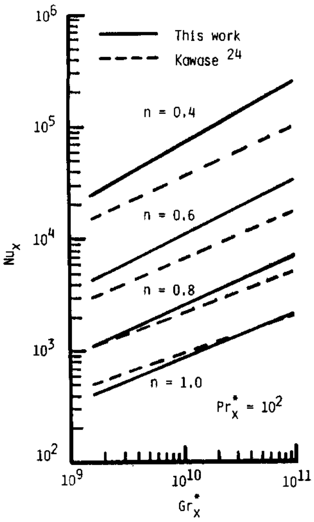

Figure 4 Comparison of the local Nusselt number from the present work with the theoretical predictions of Kawase[24] for non-Newtonian fluids at $Pr_x^* = 100$

Non-Newtonian fluids

There are no experimental data on turbulent free convection heat transfer to power-law fluids yet. This is because experimentalists shied away from this area on the assumption that it is very difficult to set up turbulent convection in power-law fluids that are known to have high viscosities. Recently, Shenoy[34] has shown that under certain criteria it is possible to theoretically set up turbulent free convection currents in commonly used viscous inelastic fluid systems. While such experimental evidence is awaited, the present analysis can only be compared with the available theoretical solution of Kawase.[24] Figure 4 shows a comparison of Equation 23 and that of Kawase[24] as given below for the vertical flat plate. Equation 23 is modified to a form congruent with the definitions used by Kawase[24] to give the following:

$$Nu_x = C\{8^{n-1}[(3n+1)/4n]^n\}^{c'} Gr_x^{*a'} Pr_x^{*b'} \tag{28a}$$

where

$$a' = \{6 + 7\beta(n-1)\}/2(n+1)\{6 - \beta(10-n)\} \tag{28b}$$

$$b' = \{3 - 7\beta(8+n)\}/\{6 - \beta(10-n)\} \tag{28c}$$

$$c' = -9\beta/\{6 - \beta(10-n)\} \tag{28d}$$

The equation proposed by Kawase[24] is given as

$$Nu_x = f(n)(1/2)(4-n)/2(n+1) Gr_x^{*(5n+7)/6(n+1)(n+2)} Pr_x^{*1/3} \tag{28e}$$

where

$$Gr_x^* = (\rho/K)^2 x^{2+n} (g_x \beta_0 \Delta T)^{2-n} \tag{29a}$$

$$Pr_x^* = \frac{\rho C p}{k} (K/\rho)^{2/(n+1)} x^{(n-1)/2(n+1)} (g_x \beta_0 \Delta T)^{3(n-1)/2(n+1)} \tag{29b}$$

$$f(n) = 0.075 n^{1/3} \{\exp(1.37n + 1.71)\}^{(4-n)/6n(n+1)} \tag{29c}$$

It can be seen from Figure 4 that the trends shown by Equations 28a and 28e are the same. The method of solution in the two cases is very different. Kawase[24] uses the energy dissipation concept and does not use any postulated temperature or velocity distributions in the theoretical analysis. The present integral method follows an entirely different line of argument to obtain the solution. Considering the diversity in the method of solutions, the agreement between the two is reasonable.

Conclusions

In the present effort, the Shenoy–Mashelkar integral approach for a flat plate has been successfully extended to the problem of turbulent free convection from curved surfaces to non-Newtonian power-law fluids. The surface wall temperature has been allowed to vary in the streamwise direction in an arbitrary fashion.

Illustrative calculations have been carried out for a flat plate, a cone pointing downward, a horizontal circular cylinder, and a sphere. Numerical values have been furnished for speedy estimation of local and average heat transfer rates. The numerical integration results on the isothermal circular cylinder and sphere reveal that an increase in the power-law index n results in the smoothing out of the peripheral distribution of local heat transfer coefficient.

Comparison of the predictions of the present analysis with existing experimental data and theoretical numerical solution for Newtonian fluids shows reasonably good agreement despite the use of the high Prandtl number approximation. For non-Newtonian fluids, no experimental data exist and hence no comparison could be made. However, the theoretical predictions of the present work were compared with the available theoretical solution for the flat plate and found to give consistent trends despite the diversity of the two methods.

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